Two-step flag manifolds and the Horn conjecture

Kevin Purbhoo

University of British Columbia

February 2, 2008

Abstract

We give a simplification of Belkale's geometric proof of the Horn conjecture. Our approach uses the geometry of two-step flag manifolds to explain the occurrence of the Horn inequalities in a very straightforward way. The arguments for both necessity and sufficiency of the Horn inequalities are fairly conceptual when viewed in this framework. We provide examples to illustrate the method of proof.

1 Introduction

1.1 General approach

Horn's conjecture [H] was originally formulated as a recursive method for solving a problem concerning the eigenvalues of Hermitian matrices. However, as a consequence of work of Klyachko [Kly], Horn's conjecture can be reformulated as saying that the non-vanishing Schubert intersection numbers for Grassmannians satisfy a certain recursion. This recursion was first proved by Knutson and Tao [KT], using combinatorial methods. Later, a geometric proof was given by Belkale [B], which was the inspiration for this paper.

For our purposes, a Schubert problem will refer to a collection of Schubert varieties $\bar{\Omega}_{\sigma_1}, \ldots, \bar{\Omega}_{\sigma_s}$, on some partial flag variety G/P. A Schubert problem is non-vanishing if the product of the corresponding cohomology classes is non-zero. Equivalently, a Schubert problem is non-vanishing if and only if the general translates $\bar{\Omega}_{\sigma_1}^{F_1}, \ldots, \bar{\Omega}_{\sigma_s}^{F_s}$ of these Schubert varieties have a non-empty generically transverse intersection. This observation allows one to study the vanishing question inside the tangent space to the G/P.

In an effort to better understand the geometry behind Horn's conjecture, we study the tangent spaces of Schubert varieties of two-step flag manifolds, and show how these are related to the problem. The two-step flag manifold

$$Fl(d, r, \mathbb{C}^n) = \{(S, V) \mid S \subset V \subset \mathbb{C}^n, \ \dim S = d, \ dim V = r\}$$

fibres over the Grassmannian $Gr(r, \mathbb{C}^n)$, with fibre $Gr(d, V)$ at the a point $V \in Gr(r, \mathbb{C}^n)$. A Schubert problem on $Gr(r, \mathbb{C}^n)$ can be "lifted" in a number of different ways to a Schubert problem on $Fl(d, r, \mathbb{C}^n)$, in such a way that a non-empty transverse intersection on $Gr(r, \mathbb{C}^n)$ lifts to a non-empty transverse intersection on $Fl(d, r, \mathbb{C}^n)$. Each way of lifting corresponds to a non-vanishing Schubert problem inside the fibre $Gr(d, V)$. However, it is sometimes possible to see that the intersection on $Fl(d, r, \mathbb{C}^n)$ is non-transverse for some trivial reasons. These trivial conditions are seen to be the Horn inequalities.

The difficult part of the Horn conjecture is to show the sufficiency of the Horn inequalities. In our approach this amounts to showing that a non-transverse intersection on $Gr(r, \mathbb{C}^n)$ lifts to something upstairs which is not only non-transverse, but non-transverse for the aforementioned trivial reasons. However, once we have all the appropriate machinery in place, this turns out to be almost as straightforward as the "easy" direction of Horn's conjecture.

The two-step flag manifolds are not a necessary part of the argument. In principle the entire argument could be formulated inside the tangent space of the Grassmannian. This would probably even lead to a shorter proof. However, it is our opinion that the geometry of the two-step flag manifold plays a fundamental role in this picture, and to undermine its role would be remiss.

Although there are a number of new ideas in this paper, we do not claim to be presenting an original or independent proof of the Horn conjecture. Our objective in writing this was to better understand the argument in [B], to simplify it in places, and to show how it relates to some of our own previous work [P1]. A few of the results have been taken directly from [B], while some of the others merely contain old ideas which have been dressed up in a new context. We will try to indicate whenever possible which ideas and results have been borrowed.

1.2 Partial flag manifolds

1.2.1 Schubert varieties and Schubert positions

Let $G = GL(n)$, and let $P \subset G$ be a parabolic subgroup, the stabiliser subgroup of some k-step flag

$$V^0 = \{0\} = V_0^0 \subsetneq V_1^0 \subsetneq \cdots \subsetneq V_k^0 \subsetneq V_{k+1}^0 = \mathbb{C}^n.$$

We will assume for later convenience that the V_i^0 are coordinate subspaces. Let $d_j = \dim V_j$. Then G/P is a partial flag manifold (of type $0 < d_1 < \cdots < d_k < n$).

Let

$$F = \{0\} = F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq F_n = \mathbb{C}^n.$$

be a full flag on C^n. Let $B(F) \subset G$ denote its stabiliser.

Let $\sigma = \sigma_1 \ldots \sigma_n$ be a string of length n, where $\sigma_l \in \{0, \ldots, k\}$, and the number of j's in this string is $d_{j+1} - d_j$. It is a basic fact that G/P has finitely many $B(F)$-orbits, and that these orbits are indexed by the set of possible σ as follows.

Define

$$\Omega_\sigma^F = \{V \in G/P \mid \sum_{j=1}^k \dim V_j \cap F_l - \dim V_j \cap F_{l-1} = \sigma_l\}.$$

Ω_σ^F is a $B(F)$-orbit, called the **Schubert cell** of σ with respect to the flag F. When the flag F is understood, or irrelevant, we shall also denote this Ω_σ. The **Schubert variety** of σ, with respect to the flag F, is its closure $\bar{\Omega}_\sigma^F$, which represents the Schubert class $S^\sigma \in H^*(G/P)$.

Example 1.1. *If the σ_l are weakly decreasing, then Ω_σ consists of a single point which is a subflag of F. If the σ_i are weakly increasing, then Ω_σ is a dense open subvariety of G/P, and $S^\sigma = 1 \in H^*(G/P)$.*

In general we can easily calculate the dimension of this orbit:

$$\dim \Omega_\sigma = \#\{l < l' \mid \sigma_l < \sigma_{l'}\}.$$

If $V \in G/P$, then the **Schubert position** of V with respect to the flag F is the unique σ such that $V \in \Omega_\sigma^F$. When we have multiple flags $F^1, \ldots, F^s$ on $\mathbb{C}^n$, the Schubert position of V will be the s-tuple $(\sigma^1, \ldots \sigma^s)$, where σ^i is the Schubert position of V with respect to F^i.

1.2.2   01-strings from σ

From σ we construct 01-*strings* in two different ways. First, for each pair (u, v), with $0 \le u < v \le k$, we define a string $\sigma(uv)$. This is constructed as follows: we delete from $\sigma = \sigma_1 \ldots \sigma_n$ every $\sigma_l \notin \{u, v\}$. What remains will be a string consisting only of 'u's and 'v's. To convert this into a 01-string, we change every 'u' to a '0' , and every 'v' to a '1' .

Example 1.2. *If $\sigma = 01312230132$, then $\sigma(13)$ is produced as follows:*

$$\sigma = 01\mathbf{3}1\mathbf{2}2\mathbf{3}0\mathbf{13}2 \mapsto \mathbf{131313} \mapsto 010101 = \sigma(13).$$

Each $\sigma(uv)$ corresponds to a Schubert cell on a different Grassmannian. In section 2.3 we investigate the significance of these $\sigma(uv)$ on a two-step flag manifold.

In a completely different spirit, we define strings $\sigma[j] = \sigma[j]_1 \ldots \sigma[j]_n$, for $1 \le j \le k$, defined as follows.

$$\sigma[j]_l = \begin{cases} 0, & \text{if } \sigma_l \le k - j \\ 1, & \text{if } \sigma_l > k - j. \end{cases}$$

These have a very natural geometric significance. The k-step flag variety has k different projections onto Grassmannians $Gr(d_j, \mathbb{C}^n)$. The image of the Schubert cell Ω_σ is projected onto $Gr(d_j, \mathbb{C}^n)$ is the Schubert cell $\Omega_{\sigma[j]}$.

3

Example 1.3. *If $\sigma = 2103210$, then*

$$\sigma[1] = 0001000$$
$$\sigma[2] = 1001100$$
$$\sigma[3] = 1101110$$

1.2.3 Notation for Grassmannians

When P is a maximal parabolic, $G/P \cong Gr(r,n)$ is a Grassmannian, and we can equivalently index the Schubert varieties and Schubert classes by partitions. Let $\Lambda(r, n-r)$ denote the set of partitions with r-parts, whose largest part is at most $n - r$, i.e.

$$\Lambda(r, n-r) = \{\lambda = \mathbf{0} \le \lambda_1 \le \cdots \le \lambda_r \le \mathbf{n-r}\}.$$

There is a simple correspondence between these partitions and 01-strings: given a partition $\lambda \in \Lambda(r, n-r)$, one can construct a string as follows.

0. Begin with the empty string and $l = 1$.
1. Append $\lambda_l - \lambda_{l-1}$ '0's .
2. If $l = n - r$ stop.
3. Append a '1' .
4. Increment l and repeat steps 1-4.

Example 1.4. $\lambda = \mathbf{0} \le 0 \le 1 \le 3 \le 3 \le \mathbf{5}$, *corresponds to the string* 101001100.

In the case of Grassmannians, we will sometimes find it more convenient to use $\Lambda(r, n-r)$ to index our Schubert classes and Schubert varieties. We therefore denote these S^λ and $\bar\Omega_\lambda$ respectively. It is worth noting at this time that

$$\dim \Omega_\lambda = \sum_{l=1}^{n-r} \lambda_l =: |\lambda|.$$

1.2.4 Induced flags

Whenever we have a full flag F on a vector space W, and $V \subset W$ is a subspace, we get induced flags F_V and $F_{W/V}$ on V and W/V respectively. These can be thought of as follows.

Consider the chain of subspaces

$$F_V = \left(\{0\} = F_0 \cap V \subset F_1 \cap V \subset \cdots \subset F_{n-1} \cap V \subset F_n \cap V = V \right).$$

Since $\dim V$ may be less than or equal to $\dim W$, this will no longer be a flag, as some of the $F_{i-1} \cap V \subset F_i \cap V$ may become equalities. However by eliminating all

repeated elements, one gets a full flag on V. It is easy to check that the elements we keep correspond precisely to the '1's in the Schubert position of V with respect to F.

Similarly we can construct a full flag on W/V. Let $\Pi : W \to W/V$ denote the quotient map. Then

$$F_{W/V} = \Big(\{0\} = \Pi(F_0) \subset \Pi(F_1) \subset \cdots \subset \Pi(F_{n-1}) \subset \Pi(F_n) = W/V \Big).$$

Again by eliminating repeated elements we obtain a full flag. In this case, we keep the elements corresponding to the '0's in the Schubert position of V.

1.3 The generalised Horn conjecture

We can now state Horn's conjecture in the language of Schubert calculus. Actually we will give the slightly more general statement which appears in [B].

To simplify the notation in the statement a little, any time we write $S^\lambda \in H^*(Gr(a,b))$ (or otherwise assume that λ indexes such a class), then we implicitly assume

$$\lambda = \mathbf{0} \leq \lambda_1 \leq \cdots \leq \lambda_a \leq \mathbf{b} - \mathbf{a} \in \Lambda(a, b-a).$$

Definition 1.5. *Let* $\lambda^1, \ldots, \lambda^s \in \Lambda(r, n-r)$ *Suppose that for every d, $1 \leq d \leq r$, and every non-zero product $S^{\mu^1} \cdots S^{\mu^s} \in H^*(Gr(d,r))$, the inequality*

$$\sum_{i=1}^{s} \sum_{k=1}^{d} \lambda^i_{\mu^i_k + k} \geq (s-1)d(n-r) \tag{1}$$

holds. In this case we say that the **Horn inequalities** *hold for* $\lambda^1, \ldots, \lambda^s$.

Theorem 1 (Generalised Horn conjecture). *The product $S^{\lambda^1} \cdots S^{\lambda^s} \in H^*(Gr(r,n))$ is non-zero if and only if the Horn inequalities hold for $\lambda^1, \ldots, \lambda^s$.*

The standard form of Horn's conjecture assumes that the product of $S^{\lambda^1} \cdots S^{\lambda^s}$ is of top degree, whereas this formulation (which appears in [B]) does not. To accommodate this, definition 1.5 allows an inequality for all non-zero products in $H^*(Gr(d,r))$ rather than those just those of top degree. This generalisation is easily shown to imply the standard statement.

Definition 1.5 differs slightly from the usual definition of the Horn inequalities in another small way. Normally one does not include an inequality for the case $d = r$; one simply uses d for which $1 \leq d \leq r - 1$. The $d = r$ inequality simply amounts to saying that

$$\sum_{j=1}^{s} \operatorname{codim} \Omega_{\lambda^i} \leq \dim Gr(r,n),$$

i.e. this covers the case where the product vanishes for dimensional reasons. If we assume that $S^{\lambda^1} \cdots S^{\lambda^s}$ is of top degree then this inequality is always satisfied.

2 Schubert calculus in the tangent space

2.1 General statements for G/P

Let $F^1, \ldots, F^s$ be flags on $\mathbb{C}^n$. To determine whether a product $S^{\sigma^1} \cdots S^{\sigma^s} \in H^*(Gr(r, n))$ vanishes, it is sufficient to consider whether the Schubert varieties $\bar{\Omega}_\sigma^{F^i}$ have a point of intersection when the flags F^i are sufficiently generic. This is due to the Kleiman-Bertini theorem [Kl], which tells us that if F^i are generic, these Schubert varieties will intersect transversely (in positively oriented points). Moreover, this point of intersection can be assumed to be inside the open cell $\Omega_\sigma^{F^i}$.

Thus we can take the following approach. Choose the flags F^i such that $V \in G/P$ is an intersection point of the Schubert varieties $\Omega_\sigma^{F^i}$. Subject to this restriction, the flags F^i should be as generic as possible. We call generic flags with this restriction **almost generic** for V. The question then becomes whether or not the Schubert varieties with respect to almost generic flags intersect transversely. If they intersect transversely, then Schubert varieties with respect to fully generic flags must have a point of intersection; however if they do not, then the point of intersection is artificial, and fully generic flags will not give any point of intersection.

More importantly, this calculation can be done on the level of tangent spaces. We have the following basic result which appears in [B, P1].

Lemma 2.1. $S^{\sigma^1} \cdots S^{\sigma^s} = 0 \in H^*(G/P)$ *if and only if the intersection of subspaces*

$$\bigcap_{i=1}^s T_V \Omega_{\sigma^i}^{F^i} \subset T_V G/P$$

is non-transverse for F^i almost generic.

Our V can be any point of G/P, so let us take $V = V^0$. Then P acts transitively on the flags F^i such that $V^0 \in \Omega_{\sigma^i}^{F^i}$. Thus to calculate $T_{V^0} \Omega_{\sigma^i}^{F^i}$, for generic F^i, it suffices to compute it for a special F^i and consider the action of a generic element of P on the tangent space $T_{V^0} G/P = \mathfrak{g}/\mathfrak{p}$.

There is always at least one coordinate flag $\hat{F}(\sigma^i)$ such that $V^0 \in \Omega_{\sigma^i}^{\hat{F}(\sigma^i)}$; we will use one of these. The standard flag on $\mathbb{C}^n$ is

$$F^{\text{std}} = \{0\} \subsetneq \langle x_1 \rangle \subsetneq \langle x_1, x_2 \rangle \subsetneq \cdots \subsetneq \langle x_1, x_2, \ldots, x_{n-1} \rangle$$

The Weyl group S_n acts transitively on the set of coordinate flags of $\mathbb{C}^n$, thus $\hat{F}^i = w_i \cdot F^{\text{std}}$ for some $w_i \in S_n$. Of all possible choices for the $\hat{F}(\sigma^i)$, we will always choose $\hat{F}(\sigma^i) = w_i \cdot F^{\text{std}}$ with w_i minimal in the Bruhat order.

Denote the tangent space to the Schubert variety $\Omega_\sigma^{\hat{F}(\sigma)}$ at V_0 by

$$\hat{Z}_\sigma = T_{V^0} \Omega_\sigma^{\hat{F}(\sigma)} \subset \mathfrak{g}/\mathfrak{p}.$$

Let Z_σ denote a generic P-translate of $\hat{Z}_\sigma$. This notation will be convenient, as we will often need to consider intesections $\bigcap_{i=1}^s p_i \hat{Z}_{\sigma_i}$ where $p_i \in P$ are generic; we can now write this simply as $\bigcap_{i=1}^s Z_{\sigma_i}$. For the special cases of Grassmannians and two-step flag manifolds, we will use the letters X or Y respectively instead of Z.

Lemma 2.1 can be stated equivalently as follows:

Lemma 2.2. $S^{\sigma^1} \cdots S^{\sigma^s} = 0 \in H^*(G/P)$ *if and only if the intersection of subspaces*

$$\bigcap_{i=1}^s Z_{\sigma^i} \subset \mathfrak{g}/\mathfrak{p}$$

is non-transverse.

Remark 2.1. *Although we have eliminated the flags F^i from this statement, we will sometimes wish to think of Z_{σ^i} as being determined by generic flags, rather than as generic translates of $\hat{Z}_{\sigma^i}$.*

Remark 2.2. *Whenever we use the notation Z_{σ^i}, we tacitly assume (unless otherwise specified) that underlying flags F^i are almost generic. If the F^i are almost generic, we say that $Z_{\sigma^1}, \ldots, Z_{\sigma^s}$ are in* **general position**. *This is, of course, equivalent to the statement that Z_{σ^i} is a generic P-translate of $\hat{Z}_{\sigma^i}$.*

To calculate a particular $\hat{Z}_\sigma$, we use the following fact.

Proposition 2.3. $\hat{Z}_\sigma$ *is the image of $\mathfrak{b}(\hat{F}(\sigma))$ under $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$.*

Proof. As $\Omega_\sigma^{\hat{F}(\sigma)}$ is a $B(\hat{F}(\sigma))$ orbit, the tangent space is generated by $\mathfrak{b}(\hat{F}(\sigma))$. $\quad\square$

2.2 Grassmannians

2.2.1 The tangent space to a Grassmannian Schubert variety

To distinguish the Grassmannian as a special case, we use the notation $\hat{X}_\lambda$ for $\hat{Z}_\sigma$, and X_λ for the generic translate X_σ, where λ is the partition corresponding to the 01-string σ. We will take V^0 to be the coordinate subspace $\langle x_{n-r+1}, \ldots, x_n \rangle \in Gr(r, n)$.

We will now identify the subspace $\hat{X}_\lambda$. Put $V = V^0$, and $Q = \mathbb{C}^n / V$. Now $\mathfrak{g}/\mathfrak{p}$ can be naturally identified with $\mathrm{Hom}(V, Q)$, so we can view $\hat{X}_\lambda$ as a set of homomorphisms $\phi : V \to Q$. Both V and Q inherit full flags from the standard flag. V inherits the flag

$$F_V^{\mathrm{std}} = \{0\} \subsetneq \langle x_{n-r+1} \rangle \subsetneq \langle x_{n-r+1}, x_{n-r+2} \rangle \subsetneq \cdots \subsetneq \langle x_{n-r+1}, \ldots, x_{n-1} \rangle \subsetneq V$$

and Q inherits the image of

$$F_Q^{\mathrm{std}} = V \subsetneq V + \langle x_1 \rangle \subsetneq V + \langle x_1, x_2 \rangle \subsetneq \cdots \subsetneq V + \langle x_1, \ldots, x_{n-r-1} \rangle \subsetneq \mathbb{C}^n$$

under the quotient map $\mathbb{C}^n \to Q$.

Proposition 2.4. *Under these identifications,*

$$\hat{X}_\lambda = \{\phi \in \mathrm{Hom}(V,Q) \mid \phi((F_V^{\mathrm{std}})_l) \subset (F_Q^{\mathrm{std}})_{\lambda_l} \; l = 1, \ldots, r\}.$$

Proof. Let σ denote the 01-string corresponding to λ, and put

$$\alpha_k = \begin{cases} \sum_{l=1}^{k} 1 - \sigma_l, & \text{if } \sigma_k = 0 \\ n - r + \sum_{l=1}^{k} \sigma_l, & \text{if } \sigma_k = 1. \end{cases}$$

So $\alpha_k \le n - r$ if $\sigma_k = 0$, and $\alpha_k > n - r$ if $\sigma_k = 1$. Then the flag $\hat{F}(\sigma)$ is

$$\hat{F}(\sigma) = \{0\} \subsetneq \langle x_{\alpha_1} \rangle \subsetneq \langle x_{\alpha_1}, x_{\alpha_2} \rangle \subsetneq \cdots \subsetneq \langle x_{\alpha_1}, \ldots, x_{\alpha_{r-1}} \rangle \subsetneq \mathbb{C}^n.$$

To see this, note that $V \cap \hat{F}(\sigma)_l$ jumps in dimension exactly when $\sigma_l = 1$, and α is the smallest permutation that accomplishes this.

By proposition 2.3, $\hat{X}_\sigma$ is is identified with $\mathfrak{b}(\hat{F}(\sigma))/\mathfrak{p}$. Let $\mathfrak{n}$ denote the orthogonal complement to $\mathfrak{p}$. Since V and $\hat{F}(\sigma)$ are coordinate flags, we have $\mathfrak{b}(\hat{F}(\sigma))/\mathfrak{p} = \mathfrak{b}(\hat{F}(\sigma)) \cap \mathfrak{n}/\mathfrak{p}$. Thus it suffices to determine $\mathfrak{b}(\hat{F}(\sigma)) \cap \mathfrak{n}$. We see that an element $\phi \in \mathfrak{n}$ preserves $\hat{F}(\sigma)$ if and only if for $j \in \{n - r + 1, \ldots, n\}$,

$$\phi(x_j) \subset \langle x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_j} \rangle$$

But since $\mathrm{Image}(\phi) \subset \langle x_1, \ldots, x_{n-r} \rangle$, this is equivalent to

$$\phi(x_j) \subset V + \langle x_1, \ldots, x_l \rangle = (F_Q^{\mathrm{std}})_l$$

where l is the number of '0's before the jth '1'. Thus $l = \lambda_j$. $\qquad\square$

Example 2.3. *In $Gr(4, \mathbb{C}^9)$, if $\lambda = \mathbf{0} \le 0 \le 1 \le 3 \le 3 \le \mathbf{5}$, then with respect to the standard coordinate bases on V and Q,*

$$\hat{X}_\lambda = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In general, if we write $\hat{X}_\lambda$ in this form, the number of $*$'s in column j will be λ_j.

The action of P on $\mathrm{Hom}(V, Q)$ is given by

$$\begin{pmatrix} A & 0 \\ C & B \end{pmatrix} \cdot \phi = A\phi B^{-1}.$$

To compute the generic intersection of two subspaces $X_\lambda \cap X_\mu$, it suffices to take $X_\lambda = \hat{X}_\lambda$, and $X_\mu = p \cdot \hat{X}_\mu$, where

$$p = \begin{pmatrix} & & & 1 & & & \\ & & \cdot^{\cdot^{\cdot}} & & & 0 & \\ & 1 & & & & & \\ & & & & & & 1 \\ & & & & & & \\ & 0 & & & \cdot^{\cdot^{\cdot}} & & \\ & & 1 & & & & \end{pmatrix}$$

This is the tangent space analogue of intersecting a Schubert variety with an opposite Schubert variety.

Example 2.4. *If $\lambda = \mathbf{0} \leq 0 \leq 1 \leq 3 \leq 3 \leq \mathbf{5}$, $\mu = \mathbf{0} \leq 3 \leq 3 \leq 3 \leq 5 \leq \mathbf{5}$, then*

$$X_\lambda \cap X_\mu = \begin{pmatrix} 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \cap \begin{pmatrix} * & 0 & 0 & 0 \\ * & 0 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

The intersection is codimension 18; however the expected codimension is $\operatorname{codim} X_\lambda + \operatorname{codim} X_\mu = 13 + 6 = 19$. *Thus $S^\lambda S^\mu = 0 \in H^*(Gr(r, n))$.*

2.2.2 X_λ as a set of homomorphisms

Let $\phi \in X_\lambda \subset \operatorname{Hom}(V, Q)$. This represents a direction in which we can perturb V and remain in the tangent space to the Schubert variety. The kernel, $\ker \phi$ represents the maximal subspace of V which is preserved by this perturbation. One of the key ideas which we take from Belkale's proof is to examine $\ker \phi$ for a generic $\phi \in \bigcap_{i=1}^s X_{\lambda^i}$. For now, however, let us restrict our attention to a single X_λ.

Let us recall that the construction of X_λ was actually a tangent space of a Schubert variety relative to some generic flag F on $\mathbb{C}^n$. Thus V carries an induced flag F_V, and Q carries an induced flag F_Q. The action of $GL(V) \times GL(Q)$ on $\operatorname{Hom}(V, Q)$ corresponds to changing these induced flags. F_V and F_Q carry all the relevant information (for our purposes) about the original flag F, and moreover, the action of $GL(V) \times GL(Q)$ on $Flags(V) \times Flags(Q)$ is transitive. Thus we shall stop thinking about F_V and F_Q as the induced flags of F, and instead think of them as an independent pair of generic flags on V and Q respectively.

Remark 2.5. *We have in fact just observed that the induced flags on V and $\mathbb{C}^n/V$ are generic, if $V \in Gr(r, n)$ is a generic point of intersection of Schubert varieties in general position.*

Let S be a subspace of V whose Schubert position relative to the flag F_V is ρ. Put $d = \dim S$. If $\phi \in \mathrm{Hom}(V, Q)$ and $\ker \phi \supset S$, then ϕ descends to a map $\tilde{\phi} \in \mathrm{Hom}(V/S, Q)$. We now consider the space

$$X_\lambda/S = \{\tilde{\phi} \in \mathrm{Hom}(V/S, Q) \mid \phi \in X_\lambda, \ \ker \phi \supset S\} = X_\lambda \cap \mathrm{Hom}(V/S, Q).$$

Lemma 2.5. $X_\lambda/S = X_{\lambda'}$, *where $\lambda' \in \Lambda(r - d, n - r)$ is the subpartition*

$$\lambda' = \mathbf{0} \leq \lambda_{k_1} \leq \cdots \leq \lambda_{k_{r-d}} \leq \mathbf{n - r}$$

where k_j are the positions of the '0's of ρ. The flag on V/S associated to $X_{\lambda'}$ is the induced flag $(F_V)_{V/S} =: F_{V/S}$.

Proof. The action of P simply gives a change of basis on V and Q. Thus it suffices to prove this in the case where $F = \hat{F}(\lambda)$, in which case $X_\lambda = \hat{X}_\lambda$ and $F_V = F_V^{\mathrm{std}}$.

We must verify that for every $\phi \in X_\lambda/S$, that $\phi((F_{V/S})_j) \subset (F_Q)_{\lambda'_j}$. However, this is straightforward, as $(F_{V/S})_j$ is the image under the quotient map of $(F_V)_{k_j}$ which maps to $(F_Q)_{\lambda_{k_j}} = (F_Q)_{\lambda'_j}$. Thus $X_\lambda/S \subset X_{\lambda'}$.

Moreover, every element of $X_{\lambda'}$ can be seen to have a unique lifting to $\{\phi \in X_\lambda \mid S \supset \ker \phi\}$. To see this, note that by a change of coordinates which preserves F_V we can make S is a coordinate subspace, in which case, this is obvious. Thus $X_\lambda/S = X_{\lambda'}$. $\qquad\square$

2.2.3 Genericity of S

The one remaining fact we will need is that when S is the special subspace $S = \ker \phi$, $\phi \in \bigcap_{i=1}^s X_{\lambda^i}$, then the flags F_S and $F_{V/S}$ are actually *generic* flags. It is a priori conceivable that by choosing this particular S (its definition involves the flags F^i), we could have undone all the genericity we had before, thereby ending up in a position where all the flags $F_{V/S}^i$ are not generic with respect to each other (for example, they could all be equal). This would be most unfortunate; however luckily it does not happen.

Let ρ^i denote the Schubert position of $S \subset V$ with respect to the flag F_V^i. Thus $S \in \bigcap_{i=1}^s \Omega_{\rho^i}^{F_V^i}$. We must show that S is in fact a generic point of this intersection of Schubert varieties. This is sufficient, since the induced flags at a generic point of an intersection of Schubert varieties are generic (c.f. remark 2.5).

The idea is to fix generic flags F_V^i, while the flags F_Q^i vary. We show that there cannot exist a subvariety $T \subset \bigcap_{i=1}^s \Omega_{\rho^i}^{F_V^i}$ such that $\ker \phi \in T$, for a generic choice of $\phi \in \bigcap_{i=1}^s X_{\lambda^i}$, and generic induced flags F_Q^i. This is a consequence of a generalisation of the Kleiman moving lemma, due to Belkale.

Lemma 2.6 (Belkale [B]). *Let H be an algebraic group acting on a variety X. Suppose $\pi : X \to Y$ is an H-invariant fibration, such that H acts transitively on the fibres. Let $Z_i \subset X$, and $Y_i = \pi(Z_i) \subset Y$ $i \in \{1, \ldots, s\}$, be subvarieties, such that $\pi|_{Z_i} : Z_i \to Y_i$ is a fibration. Put $Y_0 = \bigcap_{i=1}^{s} Y_i$, and $X_0 = \pi^{-1}(Y_0)$. Let $T \subset Y_0$, be a subvariety. Then for generic $h_i \in H$, the intersection*

$$\pi^{-1}(T) \cap \left(\bigcap_{i=1}^{s} h_i \cdot Z_i \right)$$

has the expected dimension as an intersection inside X_0. That is,

$$\operatorname{codim} \pi^{-1}(T) \cap \left(\bigcap_{i=1}^{s} h_i \cdot Z_i \right) = \operatorname{codim} \pi^{-1}(T) + \sum_{i=1}^{s} \operatorname{codim}(Z_i \cap X_0) \qquad (2)$$

(here codim *means "codimension inside X_0").*

In this example,

$$X = \operatorname{Hom}_d(V, Q) = \{\psi \in \operatorname{Hom}(V, Q) \mid \dim \ker \psi = d\},$$

$$Y = Gr(n - d, V), \qquad \pi(\psi) = \ker(\psi)$$

The group H is $GL(Q)$, which we note preserves the kernel of $\psi \in Z_i$. The action of $GL(Q)$ acts transitively on the induced flags $F^i(Q)$.

The subvariety $Z_i \subset X_{\lambda^i} \subset X$ consists of those elements of X_{λ^i} whose kernel is in Schubert position ρ^i. Thus a generic translate $h_i \cdot Z_i$ is simply X_{λ^i} for a different generic choice of flags F^i_Q. Thus the image $\pi(\psi)$ of a generic point in the intersection $\psi \in \bigcap_{i=1}^{s} h_i \cdot Z_i$, is just the kernel of a generic element of $\bigcap_{i=1}^{s} X_{\lambda^i}$.

The image of Z_i under π is the open Schubert cell $Y_i = \Omega_{\rho^i}^{F_V^i}$. We wish to show that there is no subvariety $T \subset \bigcap Y_i$ such that $\ker \psi \in T$ for all choices above. In other words, for any subspace T we can choose ψ and generic flags F^i_Q such that $\psi \in \bigcap_{i=1}^{s} X_{\lambda^i}$ and $\ker \psi \notin T$. Equivalently, we must show there exists $\psi \in \bigcap h_i \cdot Z_i$, with $\psi \notin \pi^{-1}(T)$. But this is clear from lemma 2.6, since otherwise the codimensions in equation (2) would not add.

Corollary 2.7. *Let $S = \ker \phi$ for a generic $\phi \in \bigcap_{i=1}^{s} X_{\lambda^i}$. Then the $X_{\lambda^i}/S = X_{\lambda^{i\prime}}$ are in general position. Moreover there is an element $\tilde{\phi} \in \bigcap_{i=1}^{s} X_{\lambda^{i\prime}} \subset \operatorname{Hom}(V/S, Q)$ such that $\ker \tilde{\phi} = \{0\}$.*

Proof. The fact that the X_{λ^i}/S are generic simply means that the induced flags $F^i_{V/S}$ and F^i_Q are generic, which is what we have just shown. For the second statement, since $\ker \phi = S$, ϕ descends to well defined map $\tilde{\phi} : V/S \to Q$ with $\tilde{\phi} \in \bigcap_{i=1}^{s} X_{\lambda^i}/S$ and $\ker \tilde{\phi} = \{0\}$. $\qquad \square$

Corollary 2.8. *The intersection $\bigcap_{i=1}^{s} X_{\lambda^i}/S$ is transverse.*

The argument can be seen as special case of [B][lemma 2.18].

Proof. Since the $X_{\lambda^i}/S = X_{\lambda^{i\prime}}$ are linear subspaces of $\mathrm{Hom}(V/S, Q)$ their intersection is necessarily equidimensional. Thus it suffices to show that the intersection is transverse on an open subset which contains a point of intersection.

Consider the space $\mathrm{Hom}_0(V/S, Q) = \{\psi \in \mathrm{Hom}(V/S, Q) \mid \ker \psi = \{0\}\}$. This is a homogeneous space under the action of $GL(V/S) \times GL(Q)$, and it is a Zariski open subset of $\mathrm{Hom}(V/S, Q)$. By corollary 2.7 the $X_{\lambda^{i\prime}}$ are generic translates of $\hat{X}_{\lambda^{i\prime}}$ by elements of $GL(V/S) \times GL(Q)$. So by Kleiman-Bertini, the intersection

$$\bigcap_{i=1}^{s} \left(\mathrm{Hom}_0(V/S, Q) \cap X_{\lambda^i} \right)$$

is a transverse intersection. Moreover, it contains the point $\tilde{\phi}$ which shows that the intersection $\bigcap_{i=1}^{s} X_{\lambda^{i\prime}}$ is transverse. $\qquad\square$

2.3 Two-step flag manifolds

2.3.1 The tangent space to a Schubert variety in a two-step flag manifold

Let $0 < d < r < n$. We consider the two-step flag manifold $Fl(d, r, \mathbb{C}^n)$. For two-step flag manifolds, we use the notation $\hat{Y}_\sigma$ for $\hat{Z}_\sigma$, and Y_σ for the generic translate Z_σ. Here σ is a 012-string, with d '2's , $r - d$ '1's , and $n - r$ '0's . Let $\eta_1 < \eta_2 < \cdots < \eta_{n-r}$ denote the positions of the '0's (i.e $\sigma_{\eta_m} = 0$, for $m \leq n - r$). Let $\eta_{n-r+1} < \cdots < \eta_{n-d}$ denote the positions of the '1's , and $\eta_{n-d+1} < \cdots < \eta_n$ denote the positions of the '2's .

Our first objective is to describe the space $\hat{Y}_\sigma$. Let V denote the coordinate subspace $V = \langle x_{n-r+1}, \ldots, x_n \rangle$, and S denote the coordinate subspace $S = \langle x_{n-d+1}, \ldots, x_n \rangle$. (Eventually d and S will play the same role as they did in section 2.2.) We take our base flag V^0 to be the coordinate two-step flag

$$V^0 = \{0\} \subset S \subset V \subset \mathbb{C}^n.$$

Now $\mathfrak{g}/\mathfrak{p}$ has a basis descending from the standard basis

$$\left\{ E_{jk} \mid j \leq n - r, \ k > n - d, \text{ and } (k > n - d \text{ or } j \leq n - r) \right\}$$

where E_{jk} is the image under $\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}$ of the $n \times n$ matrix whose only non-zero entry is a '1' in the (j, k)-position. These basis vectors naturally partition $\mathfrak{g}/\mathfrak{p}$ into three blocks: the upper left block is spanned by those E_{jk} such that $1 \leq j \leq n - r$, $n - r < k \leq n - d$; the lower right block is spanned by those E_{jk} such that $n - r < j \leq n - d$, $n - d < k \leq n$, and the upper right block is spanned by those E_{jk} such that $1 \leq j \leq n - r$, $n - d < k \leq n$. The first two are naturally viewed as subspaces, while the last is more naturally viewed as a quotient space.

Since S and V are coordinate subspaces, $\hat{Y}_\sigma$ is spanned by some subset of the E_{jk}.

Proposition 2.9. $\hat{Y}_\sigma = \{E_{jk} \mid \eta_j < \eta_k\}$.

Proof. $\eta = \eta_1 \ldots \eta_n$ is the element of the S_n such that $\eta^{-1} \cdot F^{\mathrm{std}} = \hat{F}(\sigma)$. Thus $\mathfrak{b}(\hat{F}(\sigma)) = \eta^{-1} \cdot \mathfrak{b}(F^{\mathrm{std}})$. One can easily check that the image in $\mathfrak{g}/\mathfrak{p}$ is $\{E_{jk} \mid \eta_j < \eta_k\}$. $\qquad\square$

Example 2.6. *Let $d = 2$, $r = 5$, $n = 9$. Then $\mathfrak{g}/\mathfrak{p}$ looks like*

$$
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & * & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

Let $\sigma = 021010201$. Then $\eta = 146835927$, and

$$
\hat{Y}_\sigma =
\begin{pmatrix}
\cdot & \cdot & \cdot & \cdot & * & * & * & * & * \\
\cdot & \cdot & \cdot & \cdot & 0 & * & * & 0 & * \\
\cdot & \cdot & \cdot & \cdot & 0 & 0 & * & 0 & * \\
\cdot & \cdot & \cdot & \cdot & 0 & 0 & * & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & * \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{pmatrix}
$$

Note that the above pictures of $\mathfrak{g}/\mathfrak{p}$ and $\hat{Y}_\sigma$ break up into three blocks,

$$
\begin{pmatrix}
* & * & * \\
0 & * & * \\
0 & 0 & * \\
0 & 0 & *
\end{pmatrix}
\qquad
\begin{pmatrix}
* & * \\
0 & * \\
0 & * \\
0 & 0
\end{pmatrix}
\qquad
\begin{pmatrix}
0 & * \\
0 & * \\
0 & 0
\end{pmatrix}
$$

and each block contains a Young-diagram shaped picture. These pictures are $\hat{X}_{\sigma(01)}$, $\hat{X}_{\sigma(02)}$ and $\hat{X}_{\sigma(12)}$.

2.3.2 Grassmannian problems in the two-step flag manifold

In example 2.6 one can see that $\mathfrak{g}/\mathfrak{p}$ consists of three rectangular blocks, and the diagram representing $\hat{Y}_\sigma$ is shaped like a Young diagram when restricted to each of these blocks. Thus these are actually diagrams for some $\hat{X}_\lambda$.

Inside $\mathfrak{g}/\mathfrak{p}$ there are two natural subspaces, corresponding to the fibres of the forgetful maps $G/P \to Gr(r, \mathbb{C}^n)$ and $G/P \to Gr(d, \mathbb{C}^n)$. The subspaces are $\mathrm{Hom}(S, V)$, and $\mathrm{Hom}(V/S, Q)$ (where $Q = \mathbb{C}^n/V$). In terms of our basis for $\mathfrak{g}/\mathfrak{p}$, these correspond to the lower right and upper left blocks respectively. Note that these subspaces are in fact P-invariant. Thus the quotient map

$$\Pi_{\mathrm{Hom}(S,Q)} : \mathfrak{g}/\mathfrak{p} \to \mathrm{Hom}(S, Q) = (\mathfrak{g}/\mathfrak{p})/\big(\mathrm{Hom}(S, V) \oplus \mathrm{Hom}(V/S, Q)\big)$$

is P-equivariant.

By simply noting that the dimensions are correct we see that is it possible to view $X_{\sigma(01)}$ as a subspace of $\mathrm{Hom}(V/S, Q)$, $X_{\sigma(12)} \subset \mathrm{Hom}(S, V)$, and $X_{\sigma(02)} \subset \mathrm{Hom}(S, Q)$. However, better than this, we have the following result.

Proposition 2.10. *We have the following identifications.*

1. *$Y_\sigma \cap \mathrm{Hom}(V/S, Q) = X_{\sigma(01)}$.*

2. *$Y_\sigma \cap \mathrm{Hom}(S, V) = X_{\sigma(12)}$.*

3. *$\Pi_{\mathrm{Hom}(S,Q)}(Y_\sigma) = X_{\sigma(02)}$.*

Moreover, if Y_{σ^i}, $i \in \{1, \ldots, s\}$ are in general position, then so are $X_{\sigma^i(01)}$, $X_{\sigma^i(12)}$, and $X_{\sigma^i(02)}$.

Proof. Since the spaces $\mathrm{Hom}(V/S, Q), \mathrm{Hom}(S, V) \subset \mathfrak{g}/\mathfrak{p}$ are P-invariant, it suffices to check statements 1–3 for $\hat{Y}_\sigma$.

For statement 1, let $\zeta_1 < \cdots < \zeta_{n-d}$, and $\zeta_{n-d+1} < \cdots < \zeta_{n-r}$ denote the positions of the '0's and '1's respectively for $\sigma(01)$, and let λ denote the partition corresponding to $\sigma(01)$. Note that $j < \lambda_k$ if and only if $\zeta_j < \zeta_k$ if and only if $\eta_j < \eta_k$. Thus we have

$$\hat{Y}_\sigma \cap \mathrm{Hom}(V/S, Q) = \langle E_{jk} \mid 1 \leq j \leq n - r,\ n - r < k < n - d,\ \text{and}\ \eta_j < \eta_k \rangle$$

On the other hand,

$$\begin{aligned}
\hat{X}_{\sigma(01)} &= \langle E_{jk} \mid E_{jk}((F_{V/S}^{\mathrm{std}})_l) \subset (F_Q^{\mathrm{std}})_{\lambda_l},\ \forall l \rangle \\
&= \langle E_{jk} \mid E_{jk}((F_{V/S}^{\mathrm{std}})_k) \subset (F_Q^{\mathrm{std}})_{\lambda_k} \rangle \\
&= \langle E_{jk} \mid (F_Q^{\mathrm{std}})_j \subset (F_Q^{\mathrm{std}})_{\lambda_k} \rangle \\
&= \langle E_{jk} \mid j < \lambda_k \rangle \\
&= \langle E_{jk} \mid \zeta_j < \zeta_k \rangle
\end{aligned}$$

To see the genericity, note that P acts on $\mathrm{Hom}(V/S, Q)$, and contains the subgroup $GL(V/S) \times GL(Q)$ (acting in the usual way). Thus if Y_{σ^i} are generic, so are $X_{\sigma^i(01)}$. A similar argument holds for $\sigma(12)$ and $\sigma(02)$. $\square$

Even more importantly, we can sometimes use the transversality (or lack thereof) for these Grassmannian Schubert problems to deduce transversality in the two-step flag manifold.

Lemma 2.11. *Let $\sigma^1, \ldots, \sigma^s$ be 012-strings, giving rise to Schubert classes on the two-step flag manifold $Fl(d, r, \mathbb{C}^n)$.*

1. *If $\bigcap_{i=1}^s X_{\sigma^i(02)}$ is non-transverse, then $\bigcap_{i=1}^s Y_{\sigma^i}$ is non-transverse.*

2. *If $\bigcap_{i=1}^s X_{\sigma^i(01)}$, $\bigcap_{i=1}^s X_{\sigma^i(12)}$, and $\bigcap_{i=1}^s X_{\sigma^i(02)}$ are all transverse, then $\bigcap_{i=1}^s Y_{\sigma^i}$ is transverse.*

Proof. These follow from elementary facts about intersections in quotient spaces and subspaces. $\qquad\square$

One important special case of this theorem is the following, which is a variant on the vanishing criterion in [P1].

Corollary 2.12. *If $\sum_{i=1}^s \dim X_{\sigma^i(02)} < (s-1)d(n-r)$ then $\bigcap_{i=1}^s Y_{\sigma^i}$ non-transverse.*

Proof. If $\sum_{i=1}^s \dim X_{\sigma^i(02)} < (s-1)d(n-r)$ then $\bigcap_{i=1}^s X_{\sigma^i(02)}$ is non-transverse for dimensional reasons. $\qquad\square$

2.3.3 The strings $\sigma[1]$ and $\sigma[2]$

We can view $\mathrm{Hom}(V, Q)$ and $\mathrm{Hom}(S, \mathbb{C}^n/S)$ as quotient spaces of $\mathfrak{g}/\mathfrak{p}$ as well: $\mathrm{Hom}(V, Q) = (\mathfrak{g}/\mathfrak{p})/\mathrm{Hom}(S, V)$ and $\mathrm{Hom}(S, \mathbb{C}^n/S) = (\mathfrak{g}/\mathfrak{p})/\mathrm{Hom}(V/S, Q)$. Let $\Pi_{\mathrm{Hom}(V,Q)}$ and $\Pi_{\mathrm{Hom}(S,\mathbb{C}^n/S)}$ denote the projection maps. Counting '1's and '0's , we see $X_{\sigma[2]}$ and $X_{\sigma[1]}$ are of the correct dimensions to view $X_{\sigma[2]} \subset \mathrm{Hom}(V, Q)$ and $X_{\sigma[1]} \subset \mathrm{Hom}(S, \mathbb{C}^n/S)$.

Proposition 2.13. *$\Pi_{\mathrm{Hom}(V,Q)}(Y_\sigma)$ is an $X_{\sigma[2]}$, and $\Pi_{\mathrm{Hom}(S,\mathbb{C}^n)}(Y_\sigma)$ is an $X_{\sigma[1]}$; however, they are not in general position.*

Proof. Let F be a flag such that $(S, V) \in \Omega_\sigma^F$. Then $V \in \Omega_{\sigma[2]}^F$, and $S \in \Omega_{\sigma[1]}^F$. It follows that $Y_\sigma = \mathfrak{b}(F)/\mathfrak{p}$ maps surjectively onto $X_{\sigma[2]} = \mathfrak{b}(F)/\mathrm{Stab}(V)$ and $X_{\sigma[1]} = \mathfrak{b}(F)/\mathrm{Stab}(S)$. The quotient maps are precisely $\Pi_{\mathrm{Hom}(V,Q)}$ and $\Pi_{\mathrm{Hom}(S,\mathbb{C}^n/S)}$.

They are not in general position, since the flag F is almost generic for the two step flag $\{0\} \subset S \subset V \subset \mathbb{C}^n$, but not necessarily almost generic for V or S. $\qquad\square$

We can now tie this to the material in section 2.2.2.

Proposition 2.14. *Fix a flag F such that $(S, V) \in \Omega_\sigma^F$. View $X_{\sigma[2]} \subset \mathrm{Hom}(V, Q)$ and $X_{\sigma(01)} \subset \mathrm{Hom}(V/S, Q)$, as being tangent spaces to Schubert varieties relative the flag F (or its induced flags). Then $X_{\sigma[2]}/S = X_{\sigma(01)}$.*

Proof. Relative to the flag F, $X_{\sigma[2]}/S = \Pi_{\mathrm{Hom}(V,Q)}(Y_\sigma) \cap \mathrm{Hom}(V/S, Q)$ and $X_{\sigma(01)} = Y_\sigma \cap \mathrm{Hom}(V/S, Q)$. These are the same as $\Pi_{\mathrm{Hom}(V,Q)}$ restricted to $\mathrm{Hom}(V/S, Q)$ is the identity map. $\qquad\square$

3 Proof of Horn's conjecture

3.1 The two-step flag manifold as a fibration

Consider the map $q : Fl(d,r,\mathbb{C}^n) \to Gr(r,\mathbb{C}^n)$ defined by $q(S,V) = V$. This is a fibration, and the fibre over V is simply the Grassmannian $Gr(d,V)$.

Schubert varieties behave well under this fibration. Let F be a flag on $\mathbb{C}^n$, and σ a 012-string representing a Schubert class on $Fl(d,r,\mathbb{C})$. Then the restriction of q to the Schubert cell Ω_σ^F is also a fibration. We have $q(\Omega_\sigma^F) = \Omega_{\sigma[2]}^F$ and the fibre over V is the Schubert cell $\Omega_{\sigma(12)}^{F_V}$. Thus from this picture, we see that intersection of Schubert varieties in $Fl(d,r,\mathbb{C}^n)$ is related to two Schubert intersection problems in Grassmannians: one on the base space $Gr(r,\mathbb{C}^n)$ and one on the fibre $Gr(d,V)$. The precise relationship is as follows:

Lemma 3.1. *Let $F^1,\ldots,F^s$ be generic flags on $\mathbb{C}^n$. Assume that $S^{\sigma^1(12)} \cdots S^{\sigma^s(12)} \neq 0 \in H^*(Gr(d,r))$. Then $\bigcap_{i=1}^s \Omega_{\sigma^i}^{F^i}$ has a point of intersection if and only if $\bigcap_{i=1}^s \Omega_{\sigma^i[2]}^{F^i}$ has a point of intersection.*

Proof. The "only if" direction is clear: if $\bigcap_{i=1}^s \Omega_{\sigma^i}^{F^i}$ is non-empty, then so is its image under q.

Thus suppose $I = \bigcap_{i=1}^s \Omega_{\sigma^i[2]}^{F^i}$ is non-empty. $S^{\sigma^1(12)} \cdots S^{\sigma^s(12)} \neq 0 \in H^*(Gr(d,r))$ is equivalent to the Schubert problem in the fibre $\bigcap_{i=1}^s \Omega_{\sigma^i(12)}^{F_V^i}$ having a point of intersection for V which induce generic flags F_V^i. But by remark 2.5, a generic $V \in I$ has this property. Thus there is a point in $\bigcap_{i=1}^s \Omega_{\sigma^i}^{F^i}$ over a generic point in I. In particular this intersection is non-empty. $\square$

This simple lemma gives us a way (in fact many different ways) of turning a Grassmannian Schubert intersection problem into a Schubert intersection problem on a two-step flag manifold. Note that given any 01-strings τ and μ with the correct number of '0's and '1's , we can always construct a 012-string σ such that $\sigma[2] = \tau$ and $\sigma(12) = \rho$.

Definition 3.1. *We call $(\sigma^1,\ldots,\sigma^s)$ a **lifting** of $(\tau^1,\ldots,\tau^s)$, if $\sigma^i[2] = \tau^i$, $\sigma^i(12) = \rho^i$ and $S^{\rho^1} \cdots S^{\rho^s} \neq 0 \in H^*(Gr(d,r))$.*

Thus we see that there is a lifting corresponding to each s-tuple $(\rho_1,\ldots,\rho_s)$ such that $S^{\rho^1} \cdots S^{\rho^s} \neq 0 \in H^*(Gr(d,r))$. If we allow d to vary between 1 and r, we also have one Horn inequality for each such triple. This is no coincidence: we are about to see that each Horn inequality arises from applying corollary 2.12 to a lifting.

3.2 Necessity of the Horn inequalities

We show that each lifting of a Grassmannian Schubert problem gives rise to a Horn inequality. This argument will establish the necessity of the Horn inequalities.

Suppose $\lambda^1, \ldots, \lambda^s \in \Lambda(r, n - r)$, with

$$S^{\lambda^1} \cdots S^{\lambda^s} \neq 0 \in H^*(Gr(r, \mathbb{C}^n))$$

Then $\bigcap_{i=1}^{s} \Omega_{\lambda^i}^{F^i}$ contains a point of intersection. Thus so does any lifting of this Schubert problem.

Let $(\sigma^1, \ldots, \sigma^s)$ be such a lifting, say, corresponding to an s-tuple of 01-strings $(\rho^1, \ldots, \rho^s)$. Let μ^i denote the partition corresponding to ρ^i. Since $\bigcap_{i=1}^{s} \Omega_{\sigma^i}^{F^i}$ is non-empty, for generic flags, the intersection of tangent spaces $\bigcap_{i=1}^{s} Y_\sigma^i$ must be transverse. By corollary 2.12 this means that

$$\sum_{i=1}^{s} \dim X_{\sigma^i(02)} \geq (s - 1)d(n - r) \tag{3}$$

is a necessary inequality.

We now compute $\dim X_{\sigma(02)}$ (for ease of notation we are fixing i and omitting the superscript i from $\sigma, \rho, \mu, \lambda$). Let $z(j)$ denote the number of '0's in the list $\sigma_1, \cdots, \sigma_{j-1}$.

$$
\begin{aligned}
\dim X_{\sigma(02)} &= \#\{(j', j) \mid j' < j, \ \sigma_{j'} = 0, \ \sigma_j = 2\} \\
&= \sum_{j \mid \sigma_j = 2} \lambda_{j - z(j)} \\
&= \sum_{k=1}^{d} \lambda_{\text{position of } k^{\text{th}} \text{ '1' in } \rho} \\
&= \sum_{k=1}^{d} \lambda_{\mu_k + k}
\end{aligned}
$$

Rewriting (3) based on this calculation, we obtain

$$\sum_{i=1}^{s} \sum_{k=1}^{d} \lambda_{\mu_k^i + k}^i \geq (s - 1)d(n - r).$$

which is exactly the inequality (1). Thus we see that the inequality (3) is precisely the Horn inequality corresponding to $(\mu_1, \ldots, \mu_s)$.

3.3 Sufficiency of the Horn inequalities

To prove sufficiency of the Horn inequalities, we must show that whenever $S^{\lambda^1} \cdots \lambda^s = 0 \in H^*(Gr(r, \mathbb{C}^n))$, there is a Horn inequality violated by the λ^i. We have already seen that a violation of the Horn inequalities gives rise to a non-transverse intersection $\bigcap_{i=1}^{s} X_{\sigma^i(02)}$ for a lifting $(\sigma^1, \ldots, \sigma^s)$ of $(\lambda^1, \ldots, \lambda^s)$. We would now like to prove the reverse: a nontransverse intersection $\bigcap_{i=1}^{s} X_{\sigma^i(02)}$ leads to a violation of a Horn inequality. This requires an inductive argument.

3.3.1 The inductive step

Lemma 3.2. *Let $\lambda^1, \ldots, \lambda^s \in \Lambda(r, n-r)$, and let $(\sigma^1, \ldots, \sigma^s)$ be a lifting of $(\lambda^1, \ldots, \lambda^s)$ to Schubert varieties in $Fl(d, r, \mathbb{C}^n)$. Let μ^i be the partition corresponding to $\sigma^i(02)$. If $\lambda^1, \ldots, \lambda^s$ satisfies all of its the Horn inequalities, then $\mu^1, \ldots, \mu^s$ must satisfy all of its Horn inequalities.*

This lemma allows us to argue by induction. Suppose the Horn conditions are sufficient for all integers d, with $d < r$. Suppose $S^{\lambda^1} \cdots S^{\lambda^s} = 0 \in H^*(Gr(r, \mathbb{C}^n))$. We show that one of the two things must be true.

1. The product is zero in cohomology for dimensional reasons.

or

2. There is a lifting of $(\lambda^1, \ldots, \lambda^s)$ to $(\sigma^1, \ldots, \sigma^s)$ such that the intersection $\bigcap_{i=1}^s X_{\sigma^i(02)}$ is non-transverse.

In the first case, the Horn inequality for $d = r$ is violated. In the second case, our inductive hypothesis tells us that some Horn inequality is violated by $\sigma^1(02), \ldots, \sigma^s(02)$. So by lemma 3.2, there must be some Horn inequality violated by $\lambda^1, \ldots, \lambda^s$.

Proof. Let $V \subset \mathbb{C}^n$, with $\dim V = r$ and fix flags $F^1, \ldots, F^s$ on $\mathbb{C}^n$ which are almost generic for V. Let $S \subset V$, with $\dim S = d$, and S in Schubert position $\sigma^i(12)$ with respect to the induced flag F_V^i.

We work inside the two-step flag manifold $Fl(d', d, V)$. let $(\chi^1, \ldots, \chi^s)$ be a lifting of $(\sigma^1(12), \ldots, \sigma^s(12))$. Of course, since this is a lifting

$$S^{\chi^1(12)} \cdots S^{\chi^s(12)} \neq 0 \in H^*(Gr(d', S)).$$

Also, the intersection of Schubert varieties $\bigcap_{i=1}^s \Omega_{\sigma^i(12)}^{F_V^i}$ is non-empty, since it contains the point S. Thus $\bigcap_{i=1}^s \Omega_{\chi^i}^{F_V^i}$ is non-empty.

Now we consider the fibration $p : Fl(d', d, V) \to Gr(d', V)$. As in the case with q, the fibration p maps Schubert varieties map to Schubert varieties. The image is $p(\Omega_\chi^F) = \Omega_{\chi[1]}^F$. Thus we see that

$$S^{\chi^1[1]} \cdots S^{\chi^s[1])} \neq 0 \in H^*(Gr(d', V)).$$

We now check that the Horn inequality for $\mu^1, \ldots, \mu^s$ corresponding to $(\chi^1(12), \ldots, \chi^s(12))$ is identical to the Horn inequality for $\lambda^1, \ldots, \lambda^s$, corresponding to $(\chi^1[1], \ldots, \chi^s[1])$. The two inequalities are

$$\sum_{i=1}^s \sum_{k=1}^d \lambda^i_{\text{position of the } k\text{th '1' in } \chi^i[1]} \geq (s-1)d'(n-r) \tag{4}$$

and

$$\sum_{i=1}^{s}\sum_{k=1}^{d} \mu^i_{\text{position of the } k^{\text{th}} \text{ '1' in } \chi^i(12)} \geq (s-1)d'(n-r). \tag{5}$$

Now, $\lambda^i_l = \mu^i_{l'}$ where $l - l'$ is the number of '0's in the list $\chi^i_1, \cdots, \chi^i_{l-1}$, thus l'. But $\chi^i[1]_l = 1$ if and only if $\chi^i_l = 2$ if and only if $\chi^i(12)_{l'} = 1$. Thus we see that

$$\lambda^i_{\text{position of the } k^{\text{th}} \text{ '1' in } \chi^i[1]} = \mu^i_{\text{position of the } k^{\text{th}} \text{ '1' in } \chi^i(12)}$$

and the two Horn inequalities (4) and (5) are the same.

Since every Horn inequality for $\mu^1, \ldots, \mu^s$ arises in this way, if $(\lambda^1, \ldots, \lambda^s)$ satisfies all its Horn inequalities, then so does $(\mu^1, \ldots, \mu^s)$. $\qquad\square$

Remark 3.2. *It is perhaps most natural to view this lemma as a statement about the three-step flag manifold $Fl(d', d, r, \mathbb{C}^n)$. We take a Schubert problem on $Gr(r, \mathbb{C}^n)$ and lift it to $Fl(d, r, \mathbb{C}^n)$. We then lift this again, to a problem on $Fl(d', d, r, \mathbb{C}^n)$, given by 0123-strings $\omega^1, \ldots, \omega^s$. We can then interpret both inequalities (4) and (5), as the statement $\sum_{i=1}^{s} \dim X_{\sigma^i(03)} \geq (s-1)d(n-r)$, which can be viewed as a Horn inequality for both λ^i and μ^i.*

3.3.2 Proof of sufficiency

We now have all the ingredients in place to prove the sufficiency of the Horn conditions.

Proof. (Horn's conjecture) As always, let $V \subset \mathbb{C}^n$, with $\dim V = r$, and fix almost generic flags F^i. Consider a generic $\phi \in \bigcap_{i=1}^{s} X^i_\lambda$. Let S be the kernel of ϕ in Schubert position ρ^i with respect to generic flags F^i_V on V. Let $(\sigma^1, \ldots, \sigma^s)$ be the lifting of $(\lambda^1, \ldots, \lambda^s)$ by $(\rho^1, \ldots, \rho^s)$.

Assume that $S^{\lambda^1} \cdots S^{\lambda^s} = 0 \in H^*(Gr(r, \mathbb{C}^n))$. Thus $\bigcap_{i=1}^{s} Y_{\sigma^i}$ is non-transverse.

There are two possibilities. Either $S = V$ or $\dim S < \dim V$. In the first case, we must have $\phi = 0$, which means that $\bigcap_{i=1}^{s} X^i_\lambda = \{0\}$ (otherwise, $\phi = 0$ would not be a generic choice). If $\sum_{i=1}^{s} \operatorname{codim} X^i_\lambda = r(n-r)$ then this is a transverse intersection, contradicting $S^{\lambda^1} \cdots S^{\lambda^s} = 0$. Thus $\sum_{i=1}^{s} \operatorname{codim} X^i_\lambda > r(n-r)$ which violates the Horn inequality for $d = r$.

If $\dim S < \dim V$, we note the following facts:

$$\bigcap_{i=1}^{s} X_{\sigma^i(12)} = \bigcap_{i=1}^{s} X_{\rho^i}$$

is a transverse intersection, since the Schubert varieties $\Omega^{F^i_V}_{\rho_i}$ are in general position, and contain S as a point of intersection. Also

$$\bigcap_{i=1}^{s} X_{\sigma^i(01)} = \bigcap_{i=1}^{s} X_{\sigma^i[1]}/S = \bigcap_{i=1}^{s} X_{\lambda^i}/S$$

19

is a transverse intersection, by corollary 2.8.

If $\bigcap_{i=1}^{s} X_{\sigma^i(02)}$ is also transverse, then by lemma 2.11, $\bigcap_{i=1}^{s} Y_{\sigma^i}$ would be transverse. Thus it is not a transverse intersection, and so by the inductive argument, some Horn inequality is violated. $\square$

3.4 Examples

The proof of sufficiency describes a procedure for finding a Horn inequality which is violated, when $S^{\lambda^1} \cdots S^{\lambda^s} = 0$. We now give some examples to show what happens in this process, in the simplest case which is $s = 2$. (The sufficiency of the Horn inequalities is an easy fact when $s = 2$; nevertheless, it illustrates the method of the proof fairly adequately.)

Example 3.3. *Let*

$$\lambda^1 = \mathbf{0} \leq 0 \leq 3 \leq 3 \leq \mathbf{4}$$

and

$$\lambda^2 = \mathbf{0} \leq 1 \leq 3 \leq 3 \leq \mathbf{4}$$

As in example 2.4, we can illustrate X_{λ^1} and X_{λ^2} in general position as

$$X_{\lambda^1} = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & * & * \\ 0 & 0 & 0 \end{pmatrix} \qquad X_{\lambda^2} = \begin{pmatrix} 0 & 0 & 0 \\ + & + & 0 \\ + & + & 0 \\ + & + & + \end{pmatrix}$$

We'll write these both in a single diagram as

Take a generic point $\phi \in X_{\lambda^1} \cap X_{\lambda^2}$, e.g.

$$\phi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The kernel of ϕ has Schubert position $(101, 101)$. We use this Schubert position to lift (λ^1, λ^2):

20

The upper right block is non-transverse for dimensional reasons. Thus we are led to consider the Horn inequality corresponding to $(101, 101)$, *i.e.*

$$\lambda_1^1 + \lambda_3^1 + \lambda_1^2 + \lambda_3^2 \geq 8.$$

The positions of the '1's in $(101, 101)$ *are the indices which appear to the indices which appear on the left hand side. We see that this inequality is violated by* (λ^1, λ^2), *as*

$$\lambda_1^1 + \lambda_3^1 + \lambda_1^2 + \lambda_3^2 \geq 0 + 3 + 1 + 3 < 8.$$

Example 3.4. *Let*
$$\lambda^1 = \mathbf{0} \leq 0 \leq 2 \leq 3 \leq 3 \leq 3 \leq 4 \leq \mathbf{4}$$

and
$$\lambda^2 = \mathbf{0} \leq 1 \leq 1 \leq 3 \leq 3 \leq 3 \leq 3 \leq \mathbf{4}.$$

We illustrate X_{λ^1} *and* X_{λ^2} *as:*

Let $\phi \in X_{\lambda^1} \cap X_{\lambda^2}$ *be a generic element, e.g.*

$$\phi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 6 & 7 & 0 & 0 \\ 0 & 0 & 8 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The kernel of ϕ *has Schubert position* $(100110, 010011)$. *We use this Schubert position to lift* (λ^1, λ^2):

The upper left, and lower right blocks have a transverse intersection. However, the upper right block

21

does not. This block represents $X_{\lambda^{1'}} \cap X_{\lambda^{2'}}$ for

$$\lambda^{1'} = \mathbf{0} \le 0 \le 3 \le 3 \le \mathbf{4}$$

$$\lambda^{2'} = \mathbf{0} \le 1 \le 3 \le 3 \le \mathbf{4}$$

In example 3.3 we found that the Horn inequality $\lambda^{1'}_1 + \lambda^{1'}_3 + \lambda^{2'}_1 + \lambda^{2'}_3 \ge 8$, which corresponds to $(101, 101)$ is violated. To find a corresponding Horn inequality which is violated by (λ^1, λ^2) we lift the Schubert position of $\ker \phi$ by $(101, 101)$ to get $(200120, 020012)$. The positions of the '2's give the indices which appear in the relevant inequality. In this case we find that the Horn inequality

$$\lambda^1_1 + \lambda^1_5 + \lambda^2_2 + \lambda^2_6 \ge 8$$

is violated. Indeed

$$\lambda^1_1 + \lambda^1_5 + \lambda^2_2 + \lambda^2_6 = 0 + 3 + 1 + 3 < 8.$$

References

[B]   P. Belkale, *Geometric Proofs of Horn and Saturation Conjectures*, J. Algebraic Geometry **15** (2006), no. 1, 133-173.

[F1] W. Fulton, *Eigenvalues of sums of Hermitian matrices.* Séminaire Bourbaki. Vol. 1997/98. Astérisque No. 252 (1998), Exp. No. 845, 5, 255–269.

[F2] W. Fulton, *Young Tableaux with Applications to Representation Theory and Geometry,* Cambridge U.P., New York, 1997.

[F3] W. Fulton, *Eigenvalues, invariant factors, highest weights, and Schubert calculus*, Bull. Amer. Math. Soc. 37 (2000), pp 209-250.

[H]   A. Horn, *Eigenvalues of sums of of Hermitian matrices*, Pacific J,. Math 12 (1962), pp. 225-241.

[Kly] A. Klyachko, *Stable bundles, representation theory, and Hermitian operators*, Institute Mittag-Leffler Preprint 1996-7; earlier version, University of Marne-la-Valée, 1994.

[Kl] S. Kleiman, *The transversality of a general translate*, Compositio Math. 28 (1974), 287-297.

[KT] A. Knutson, T. Tao, *The honeycomb model of $GL_n(\mathbb{C})$ tensor products I: proof of the saturation conjecture.* J. Amer. Math. Soc. 12 (1999), no. 4, 1055–1090.

[P1] K. Purbhoo, *Vanishing and non-vanishing criteria in Schubert calculus*, to appear in Int. Math. Res. Not.

[P2] K. Purbhoo, *Root games on Grassmannians*, preprint `math.CO/0310103`.